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The problem of determining the order to the stress singularity for an anti-plane shear crack at the interface of two bonded inhomogeneous elastic materials is considered. It had been conjectured previously that the usual square roof singularity occurs if the spatially varying material properties are merely continuous across the bond line but not necessarily continuously differentiable. This paper presents a proof of that conjecture.

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ON THE ORDER OF THE STRESS SINGULARITY FOR AN ANTI-PLANE SHEAR CRACK AT THE INTERFACE OF TWO BONDED INHOMOGENEOUS ELASTIC MATERIALS

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AND
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On the Order of the Stress Singularity for an Anti-Plane Shear Crack at the Interface of Two Bonded Inhomogeneous Elastic Materials

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1. Introduction

A number of investigations of crack problems in nonhomogeneous media have been undertaken in which the elastic moduli vary continuously with spatial coordinates. In all of these studies special forms of inhomogeneities have been assumed in order to insure a tractable problem for which the asymptotic form of the stress field near the crack tip could be calculated. One of the primary objectives of these works was to determine the effect of spatial inhomogeneity upon the known singular field quantities associated with the corresponding homogeneous problem.

In the case of a Mode I crack embedded in a nonhomogeneous medium, symmetric about the plane of the crack, specific models of inhomogeneities examined thus far show the usual square root singular crack tip stress field associated with the homogeneous medium occurs for the nonhomogeneous problem provided the shear modulus does not vanish in the plane of the crack. (see, e.g. Delale and Erdogan (1983), Gerasoulis and Srivastav (1980), Schovanec (1986), and Schovanec and Walton.)

Studies of a Mode III crack located on the interface of two bonded materials have shown that the assumption of spatial inhomogeneity in either component part does not alter the known square root singular behavior present in the piecewise homogeneous bimaterial problem. One of the more general models utilized, which includes the homogeneous and bimaterial mediums as special cases, was investigated by Delale (1985) who took the shear modulus in the form $\mu(x,y) = \mu^{\pm} e^{\alpha x+\beta^{\pm} y}$ with the crack situated in the plane $y=0$ and μ^{+}, $\beta^{+}(\mu^{-},\beta^{-})$ constants specified in the half plane $y>0$ $(y<0)$. (See also Clements, et al. (1978) and Dhaliwal and Singh (1978).)

In the situation that the crack tip terminates at a bimaterial interface between two homogeneous mediums it is known that the stress field singularity is of the form $r^{-\gamma}$, $0 < \gamma < 1$, where r is the distance from the crack tip. (Erdogan and Cook (1972)). As discussed by Atkinson (1977) such a result is unsatisfactory from a fracture mechanics point of view in that it results in a stress intensity factor and energy release rate of zero or infinity. To alleviate this inadequacy in the 'ideal' interface model Atkinson introduced a nonhomogeneous layer at the interface in which the shear modulus varied continuously, matching the constant values at the two outer mediums. For the specific forms of the modulus considered it was then shown that the stress at the crack tip displayed an $r^{-1/2}$ singularity. Assuming that the elastic module could be expanded in a Taylor series about $r=0$, Atkinson suggested that the square root behavior would always result. The antiplane shear crack perpendicular to an interface was also considered by Erdogan (1985). To investigate the effect of a continuous but nondifferentiable elastic modulus at the interface, a shear modulus of the form $\mu(x) = \mu_0 e^{\beta^{\pm} x}$ was assumed with $\beta^+(\beta^-)$ corresponding to $x>0$ $(x<0)$. The main focus of that study was to show that for such a model a square root singularity at the crack tip resulted. Erdogan proposed that the same conclusion should be valid for any continuous but nondifferentiable modulus $\mu(x)$.

This note addresses the conjecture raised in the two previous studies. Without assuming any particular form of the shear modulus, only that it be continuous throughout the medium and differentiable everywhere except along a curve at which the crack tip terminates, it is shown that a square root

singular stress field is present at the crack tip. As will be clear from the subsequent analysis, the situation in which the crack lies along this interface at which the modulus is nondifferentiable is handled in the same manner, so that in such cases the same square root behavior occurs. This note does not address the general question of existence of solutions for such a class of boundary value problems. Rather, the approach taken here is to show that when there exists a physically meaningful solution, a notion to be made precise later, then that solution exhibits a square root singular stress field.

2. The Boundary value problem.

The specific problem considered here is that of an infinite nonhomogeneous isotropic elastic solid containing a semi-infinite mode III crack terminating along a curve C corresponding to the interface of two bonded materials. Referred to cartesian coordinates x,y,z, the crack is assumed to be in the plane $y=0$, $x<0$ and the nonzero component of displacement $\omega(x,y)$ is related to the nonzero components of stress by $\sigma_{xz} = \mu\partial\omega/\partial x$, $\sigma_{yz} = \mu\partial\omega/\partial y$ where the shear modulus $\mu(x,y)$ is assumed to be continuous throughout the medium and differentiable everywhere but along C. The primary case to keep in mind is when the crack is perpendicular to the interface in which case C corresponds to $x=0$. If the modulus is symmetric about the plane of the crack then the particular boundary value problem to be solved is

$$\mathrm{div}(\mu\nabla\omega) = 0 \qquad |x|<\infty,\ y>0$$

$$\mu(x,0)\,\frac{\partial\omega}{\partial y}(x,0) = S(x) \qquad x<0$$

$$\omega(x,0) = 0 \qquad x>0.$$

The existence of a solution with locally finite strain energy, i.e., $\int_{\Omega}(\mu/2)\ |\nabla\omega|^2\ dA < \infty$ where Ω is any bounded measurable subset of the upper half-plane, will be assumed. Note that a sufficient condition to insure ω have locally finite strain energy is that near the crack tip, $|\nabla\omega| = O(r^{-\gamma})$, $0 \leq \gamma < 1$. The point here is to show $\gamma = 1/2$. A growth condition on ω of the form $\int_{|x|>R} \log|\underset{\sim}{x}|((\nabla\mu\cdot\nabla\underset{\sim}{\omega})/\mu)(\underset{\sim}{x})\ d\underset{\sim}{x} < \infty$ for $R>0$ is also assumed. For a large class of moduli, (e.g., $\mu(x) = (\alpha x+\beta)^n$ as considered in Atkinson (1977)) this condition is satisfied if $|\nabla\omega| = O(r^{-1-\varepsilon})$, $r\to\infty$, which, is indeed the case if the applied tractions are statically self-equilibrating (Cook and Erdogan (1977)). Again it should be mentioned that the question of determining conditions on the applied traction $S(x)$ and the curve C which assure the existence of such a solution ω is not taken up here.

Define the Green's function by

$$\Gamma(r,s;x,y) = \frac{1}{2\pi} \log \sqrt{[(r-x)^2 + (s-y)^2]\ [(r-x)^2 + (s+y)^2]} + \psi(r,s;x,y)$$

where

$$\frac{\partial^2\psi}{\partial r^2} + \frac{\partial^2\psi}{\partial s^2} = 0, \qquad |r|<\infty,\ s>0$$

$$\psi(r,0) = \frac{-1}{2\pi} \log[(r-x)^2 + y^2] \qquad r>0 \tag{1}$$

$$\frac{\partial\psi}{\partial s}(r,0) = 0 \qquad r<0. \tag{2}$$

Applying Green's second identity

$$\int_{\Omega} v_1 \Delta v_2 - v_2 \Delta v_1 dA = \int_{\partial\Omega} v_1 \frac{\partial v_2}{\partial n} - v_2 \frac{\partial v_1}{\partial n} ds$$

in each medium determined by the interface C with $v_1 = \omega$, $v_2 = \Gamma$, and recalling that the displacements are continuous at C while the normal tractions across C cancel, there results the representation formula

$$\omega(x,y) = -\int_{-\infty}^{\infty} \int_{0}^{\infty} \Gamma(s,r;x,y) \left(\frac{\nabla\mu \cdot \nabla\omega}{\mu}\right)(r,s)\, dsdv + \int_{-\infty}^{0} \Gamma(r,0;x,y) \frac{s(r)}{\mu(r,0)}\, dr. \tag{3}$$

With the above assumptions on ω the first integral exists and the subsequent expressions to be derived from it are also well defined.

Seek the harmonic function ψ in the form

$$\psi(r,s;x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t;x,y) \frac{s}{(t-r)^2 + s^2}\, dt.$$

Since

$$\frac{\partial\psi}{\partial s}(r,0;s,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\partial f}{\partial t}(t;x,y) \frac{dt}{t-r},$$

inversion of this relation and use of (2) and (1) gives

$$-\frac{\partial f}{\partial r}(r;x,y) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\partial\psi}{\partial s}(t,0;x,y) \frac{dt}{t-r}, \quad -\infty<r<\infty \tag{4}$$

and

$$\frac{(r-x)}{(r-x)^2+y^2} = \int_{0}^{\infty} \frac{\partial\psi}{\partial s}(t,0;x,y) \frac{dt}{t-r}, \quad r>0. \tag{5}$$

Methods for solving (5) are well known and it can be shown that

$$\frac{\partial\psi}{\partial s}(t,0;x,y) = \frac{-1}{\pi^2\sqrt{t}} \int_{0}^{\infty} \frac{(u-x)\sqrt{u}}{(u-x)^2+y^2} \frac{du}{u-t}, \quad t>0. \tag{6}$$

Substitution of (6) into (4), an interchange in the order of integration and the observation that for $u>0$, $r<0$

$$\int_0^\infty \frac{1}{\sqrt{t}} \frac{dt}{(u-t)(t-r)} = \frac{\pi}{\sqrt{|r|}} \frac{1}{u-r}$$

leads to

$$\frac{\partial f}{\partial r}(r;x,y) = \frac{1}{\pi^2\sqrt{|r|}} \int_0^\infty \frac{\sqrt{u}\,(u-x)}{(u-x)^2+y^2} \frac{du}{u-r}, \qquad r<0.$$

An integration by parts gives

$$\frac{\partial f}{\partial r}(r;x,y) = \frac{1}{4\pi^2\sqrt{|r|}} \int_0^\infty \frac{(u+r)}{\sqrt{u}\,(u-r)^2} \log[(u-x)^2+y^2]\, du, \qquad r<0. \tag{7}$$

At this point it is convenient to observe that if (7) is differentiated with respect to y and then integrated with respect to r, keeping in mind that $f(\cdot,x,y)$ must vanish at infinity, then one has for $r<0$.

$$\frac{\partial f}{\partial y}(r;x,0) = \begin{cases} -\dfrac{1}{\pi} \dfrac{1}{\sqrt{x}} \dfrac{\sqrt{|r|}}{x-r} & x>0 \\ 0 & x<0 \end{cases}. \tag{8}$$

In order to illustrate the singular nature of the stress field and to determine the stress intensity factor, only $\partial\omega/\partial y\,(x,0)$ is required. The method of computing σ_{xz} is carried out in the same manner.

3. The stress singularity and the SIF.

From the definition of Γ, (1), and the representation of ω given by (3), one has that

$$\frac{\partial \omega}{\partial y}(x,0) = \lim_{y\to 0^+} \left\{\int_{-\infty}^{\infty} \int_0^\infty \left\{\int_{-\infty}^{\infty} \frac{\partial f}{\partial y}(t;x,y) \frac{s}{(t-r)^2+s^2}\, dt\right\} \left(\frac{\nabla\mu\cdot\nabla\omega}{\mu}\right)(r,s)\, ds\,dr\right. \tag{9}$$

$$+ \frac{1}{\pi}\int_{-\infty}^{0} \frac{y}{(r-x)^2+y^2}\,\frac{s(r)}{\mu(r,0)}\,dr + \int_{-\infty}^{0} \frac{\partial f}{\partial y}(r;x,y)\,\frac{s(r)}{\mu(r,0)}\,dr\Big\} \;.$$

Making use of (8) it easily follows that the limit of the single integrals in (9) is given by

$$\begin{cases} \dfrac{s(x)}{\mu(x,0)} & x<0 \\[2ex] \dfrac{1}{\pi}\,\dfrac{1}{\sqrt{x}}\displaystyle\int_{-\infty}^{0} \sqrt{|r|}\,\frac{s(r)}{u(r)}\,\frac{dr}{r-x} & x>0. \end{cases} \tag{10}$$

By utilizing (8) and (1) it can be shown in a similar manner that the limit of the triple integral in (9) is

$$\begin{cases} \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty}\int_{0}^{\infty}\Big\{\frac{s}{(x-r)^2+s^2} + \frac{1}{\pi}\,\frac{1}{\sqrt{x}}\int_{-\infty}^{0}\frac{\sqrt{|t|}}{x-t}\,\frac{s}{(t-r)^2+s^2}\,dt\Big\}\,\frac{\nabla\mu\cdot\nabla\omega}{\mu}\,ds\,dr, & x>0 \\[2ex] 0 & x<0. \end{cases} \tag{11}$$

Combining (10) and (11) it follows that for $x>0$,

$$\sigma_{yz}(x,0) = \frac{\mu(x,0)}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{s}{(x-r)^2+s^2}\,\frac{\nabla\mu\cdot\nabla\omega}{\mu}\,ds\,dr \; + \tag{12}$$

$$\frac{\mu(x,0)}{\pi\sqrt{x}}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{0}\frac{\sqrt{|t|}}{x-t}\,\frac{s}{(t-r)^2+s^2}\,\frac{\nabla\mu\cdot\nabla\omega}{\mu}\,dt\,ds\,dr + \frac{\mu(x,0)}{\pi\sqrt{x}}\int_{-\infty}^{0}\sqrt{|r|}\,\frac{s(r)}{\mu(r,0)}\,\frac{dr}{r-x} \;.$$

From (12) the square root singular behavior is apparent and in the case that μ is constant, the known result for the homogeneous medium (Willis (1967)),

$$\sigma_{yz}(x,0) = \frac{1}{\pi\sqrt{x}}\int_{-\infty}^{0}\sqrt{|r|}\;s(r)\,\frac{dr}{r-x}$$

is easily recovered. Since a square root singularity is always present at the crack tip, the stress intensity factor may be defined as

$K = \lim_{x \to 0^+} \mu(x,0) \sqrt{x} \frac{\partial \omega}{\partial y}(x,0)$ and from (12) one has

$$K = \frac{-\mu_0}{\pi} \left\{ \int_{-\infty}^{0} \frac{1}{\sqrt{|r|}} \frac{s(r)}{\mu(r,0)} dr + \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|t|}} \frac{s}{(t-r)^2+s^2} \frac{\nabla\mu \cdot \nabla\omega}{\mu} dt ds dr \right\} .$$

As pointed out above, the calculation of σ_{xz} is carried out in a similar manner and by changing to polar coordinates it is not difficult to show that in general $\sigma_{ij}(r,\theta) = O\left(\frac{1}{\sqrt{r}}\right)$, $r \to 0$. Finally it should be noted that a similar derivation can be carried out for finite domains. In this case the appropriate boundary conditions must be prescribed and the analogous Green's function computed. The growth condition imposed on ω however would no longer be necessary.

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